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Precision predictions for the $t\bar{t}$ production cross section at hadron collidersValentin Ahrens,¹ Andrea Ferroglia,² Matthias Neubert,¹ Ben D. Pecjak,¹ and Li Lin Yang³¹*Institut für Physik (THEP), Johannes Gutenberg-Universität, D-55099 Mainz, Germany*²*New York City College of Technology, 300 Jay Street, Brooklyn, NY 11201, USA*³*Institute for Theoretical Physics, University of Zürich, CH-8057 Zürich, Switzerland*

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We make use of recent results in effective theory and higher-order perturbative calculations to improve the theoretical predictions of the top-quark pair production cross section at hadron colliders. In particular, we supplement the fixed-order NLO calculation with higher-order corrections from soft gluon resummation at NNLL accuracy. Uncertainties due to power corrections to the soft limit are estimated by combining results from single-particle inclusive and pair invariant-mass kinematics. We present our predictions as functions of the top-quark mass in both the pole scheme and the $\overline{\text{MS}}$ scheme. We also discuss the merits of using threshold masses as an alternative, and calculate the cross section with the top-quark mass defined in the 1S scheme as an illustrative example.

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I. INTRODUCTION

The total $t\bar{t}$ production cross section is an important observable at hadron colliders such as the Tevatron and LHC. For instance, it provides information about the top-quark mass, which is an input for electroweak fits [1] used to constrain the mass of the Higgs boson. Extractions of the top-quark mass from the production cross section have the advantage that the perturbative calculations used in the analysis are carried out in a well-defined renormalization scheme for the top-quark mass. The pole mass as well as the $\overline{\text{MS}}$ mass have already been extracted from the production cross section at the Tevatron [2]. The use of a short-distance mass such as the $\overline{\text{MS}}$ mass is theoretically favored over the pole mass, which can be defined only up to a renormalon ambiguity of order Λ_{QCD} . Moreover, it has been proposed in [3] that the apparent convergence and scale uncertainties of the perturbative series for the total cross section is improved in the $\overline{\text{MS}}$ scheme compared to the pole scheme, already at low orders in the perturbative expansion.

According to the QCD factorization theorem [4], the hadronic cross section can be obtained from the partonic one after convolution with parton distribution functions (PDFs). For $t\bar{t}$ production, it is conventional to express this factorization theorem in the form

$$\sigma(s, m_t) = \frac{\alpha_s^2(\mu_f)}{m_t^2} \sum_{i,j} \int_{4m_t^2}^s \frac{d\hat{s}}{s} \mathcal{F}_{ij}\left(\frac{\hat{s}}{s}, \mu_f\right) f_{ij}\left(\frac{4m_t^2}{\hat{s}}, \mu_f\right), \quad (1)$$

where μ_f is the factorization scale, $i, j \in \{q, \bar{q}, g\}$, and the parton luminosities are defined as

$$\mathcal{F}_{ij}(y, \mu_f) = \int_y^1 \frac{dx}{x} f_{i/N_1}(x, \mu_f) f_{j/N_2}(y/x, \mu_f). \quad (2)$$

Here and throughout this letter the running coupling is defined in the $\overline{\text{MS}}$ scheme with five active (massless) flavors. The scaling functions f_{ij} are proportional to the partonic cross sections and can be expanded in powers of α_s/π . Numerical results for the next-to-leading order (NLO) term have been known for over two decades [5–7], and more recently fully analytic results were obtained in [8]. Predictions based on NLO calculations typically exhibit scale uncertainties larger than 10%. In order to further reduce the theoretical uncertainties to match the experimental precision, it is necessary to go beyond NLO. Consequently, the calculation of the next-to-next-to-leading order (NNLO) corrections is an active area of research. Many efforts have been made in the calculation of two-loop virtual corrections [9–14], one-loop interference terms [15–17], and double real emission [18–21]. Due to very recent progress in developing a new subtraction scheme for double real emission in the presence of massive particles [22–26], a calculation of the NNLO corrections now seems feasible, and its completion will be a major accomplishment.

An important way to improve on the fixed-order calculations is to supplement them with threshold resummation [27, 28]. In such an approach, which in general works at the level of differential cross sections, one identifies a partonic threshold parameter which vanishes in the limit where extra real radiation is soft, and sums a certain tower of logarithmic corrections in this parameter to all orders in the strong coupling. Such resummed formulas neglect power corrections which vanish when the threshold parameter goes to zero, but these can be taken into account, up to a given order in the strong coupling, by matching with the fixed-order results. In this way, one obtains predictions which not only make full use of the fixed-order calculations, but also resum a class of logarithmic corrections to all orders. In

the limit where the higher-order corrections are dominated by these logarithmic terms, such a resummation is clearly an improvement. For this reason, many different implementations of threshold resummation have been considered in the literature, where the current frontier is next-to-next-to-leading logarithmic (NNLL) accuracy [3, 29–35].

The purpose of this letter is to consolidate results for the total cross section based on threshold resummation in effective field theory. These are obtained from two different threshold limits for the differential cross section. The first, referred to as pair invariant-mass (PIM) kinematics, uses the top-pair invariant-mass distribution as the fundamental observable [32, 33]. The second, referred to as single-particle inclusive (1PI) kinematics, works at the level of the transverse-momentum or rapidity distributions of the top quark [35]. The total partonic cross section is obtained by integrating the resummed distributions over the appropriate phase space. In both kinematics, the predictions in the effective field-theory framework include resummation effects to NNLL order, and are matched with the fixed-order results at NLO in order to achieve NLO+NNLL accuracy for the total cross section. Such formulas can be evaluated numerically using specific values of the matching scales appearing in the effective-theory calculations, or otherwise re-expanded in a fixed-order expansion, in the form of approximate NNLO predictions. We have considered both scenarios in [33, 35], finding that when the results in PIM and 1PI kinematics are combined as in the present work, the numerical differences between the NLO+NNLL and approximate NNLO results are rather small. In this letter we focus on the approximate NNLO results for concreteness.

In what follows, we briefly review the formalism, and then give our best numerical predictions for the total cross section in the pole scheme for the top-quark mass, in the form of numerical fits as functions of m_t . We then explain how to convert the results to the $\overline{\text{MS}}$ scheme and present numerical results as a function of the $\overline{\text{MS}}$ mass. The results in both schemes are calculated with a Fortran implementation of our approximate NNLO formulas, which we include with the electronic submission of this letter. Contrary to [3], we do not find a strong improvement of the perturbative series in the $\overline{\text{MS}}$ scheme compared to the pole scheme. We suggest that the group of short-distance masses known as threshold masses (see e.g. [36]) may actually be the more appropriate choice, and we explicitly consider the case of the mass defined in the 1S scheme [37]. However, we believe that the poor large-order behavior of the perturbative series in the pole scheme is unlikely to be of practical importance in the foreseeable future.

II. INGREDIENTS OF THE CALCULATION

Our calculation is based on [33, 35], where threshold resummation using soft-collinear effective theory (SCET) [38–40] is applied to $t\bar{t}$ production at hadron colliders. The partonic scattering process is

$$i(p_1) + j(p_2) \rightarrow t(p_3) + \bar{t}(p_4) + \hat{X}(k), \quad (3)$$

where i, j indicate the incoming partons and \hat{X} is a partonic final state. In [33], the invariant-mass distribution $d\sigma/dM$ of the $t\bar{t}$ pair was considered, where $M^2 \equiv (p_3 + p_4)^2$, and the threshold region is defined by $z = M^2/\hat{s} \rightarrow 1$ with $\hat{s} = (p_1 + p_2)^2$ (so-called PIM kinematics). In [35], on the other hand, the transverse-momentum and rapidity distributions of the top-quark were considered. In this case, the threshold region is defined by $s_4 = (p_4 + k)^2 - m_t^2 \rightarrow 0$ (1PI kinematics). In both the PIM and 1PI threshold limits, the energy of the extra emitted gluons vanishes, and the partonic differential cross sections are dominated by singular distributions of the form

$$\alpha_s^n \left[\frac{\ln^m \xi}{\xi} \right]_+; \quad m = 0, \dots, 2n - 1, \quad (4)$$

where $\xi = (1 - z)/\sqrt{z}$ for PIM and $\xi = s_4/(m_t \sqrt{m_t^2 + s_4})$ for 1PI kinematics.

In the threshold limit $\xi \rightarrow 0$, the partonic differential cross section can be factorized into a product of matrix-valued hard and soft functions,

$$d\hat{\sigma}(\xi, \mu_f) \propto \text{Tr} \left[\mathbf{H}(\mu_f) \mathbf{S} \left(\frac{E_s(\xi)}{\mu_f}, \alpha_s(\mu_f) \right) \right], \quad (5)$$

where we have suppressed the dependence on the other kinematic variables, and $E_s(\xi)$ is the energy of the soft gluon radiation, which is given by $E_s = M\xi$ in the partonic center-of-mass frame for PIM kinematics, and $E_s = m_t \xi$ in the \bar{t} rest frame for 1PI kinematics. The hard function \mathbf{H} is related to virtual corrections and is thus evaluated at $\xi = 0$. The ξ dependence of the cross section is encoded in the soft function \mathbf{S} via the ratio $E_s(\xi)/\mu_f$. One can therefore use the renormalization-group (RG) equation of the soft function to resum the singular distributions in ξ to all orders in α_s . A technical complication is that the RG equation of the soft function is non-local. We solve this equation using

the technique of Laplace transforms [41], and then carry out the inverse Laplace transform analytically to obtain expressions in momentum space. The final result for the resummed cross section reads

$$d\hat{\sigma}(\xi, \mu_f) \propto \exp[4a_{\gamma\phi}(\mu_s, \mu_f)] \text{Tr} \left[\mathbf{U}(\mu_h, \mu_s) \mathbf{H}(\mu_h) \mathbf{U}^\dagger(\mu_h, \mu_s) \tilde{\mathbf{s}}(\partial_\eta, \alpha_s(\mu_s)) \right] \frac{e^{-2\gamma_E\eta}}{\Gamma(2\eta)} \xi^{-1+2\eta}, \quad (6)$$

where $\tilde{\mathbf{s}}$ is the Laplace transform of the soft function, \mathbf{U} is an evolution matrix which resums large logarithms between the hard and soft scales μ_h and μ_s , and $\eta = 2a_\Gamma(\mu_s, \mu_f)$ and $a_{\gamma\phi}(\mu_s, \mu_f)$ are related to certain integrals over anomalous-dimension functions. Explicit expressions for these objects in RG-improved perturbation theory can be found in [33]. For NNLL accuracy, one needs the anomalous-dimension matrices for the hard and soft functions to two-loop order (computed in [42]), as well as the hard and soft functions to one-loop order (computed in [33, 35]). One must also specify a procedure for choosing the hard and soft scales. In the SCET approach, these are chosen such that the contributions of the NLO matching functions to the hadronic cross section are minimized; details can be found in [33, 35].

The effective theory is a powerful tool for separating physics at the formally very different scales μ_h and μ_s and for summing logarithms of their ratio. In practice, however, in areas of phase space where the differential cross section is largest, the numerical hierarchy between the two scales is only moderate. Therefore, an alternative to using the NLO+NNLL formulas directly is to re-expand them in a fixed-order series in $\alpha_s(\mu_f)$, constructing what are referred to as approximate NNLO formulas. The full NNLO correction to the cross section takes the form

$$d\hat{\sigma}^{(2)}(\xi, \mu_f) \propto \alpha_s^4(\mu_f) \left(\sum_{m=0}^3 D_m \left[\frac{\ln^m \xi}{\xi} \right]_+ + C_0 \delta(\xi) + R(\xi) \right). \quad (7)$$

The approximate NNLO formulas derived from the NLO+NNLL results contain an unambiguous answer for the D_m coefficients in the limit $\xi \rightarrow 0$. Only parts of the coefficient C_0 are determined, and the regular piece R is subleading in the threshold limit and not determined. There are thus ambiguities in what to include in the C_0 term, and a freedom to shuffle terms between the D_i and R away from the exact limit $\xi \rightarrow 0$. For our method of dealing with these ambiguities, we refer the reader to [35], which specifies all details relevant for our implementation of the approximate NNLO formulas used in the numerical analysis.

Equations (1) and (5)–(7) are written for the case where the factorization and renormalization scales are set equal, $\mu_f = \mu_r$. Later on we will consider the case where the running coupling is instead evaluated at an arbitrary renormalization scale μ_r . To derive such expressions we use

$$\alpha_s(\mu_f) = \alpha_s(\mu_r) \left[1 + \frac{\alpha_s(\mu_r)}{\pi} \frac{23}{12} L_{rf} + \left(\frac{\alpha_s(\mu_r)}{\pi} \right)^2 \left(\frac{529}{144} L_{rf}^2 + \frac{29}{12} L_{rf} \right) \right], \quad (8)$$

with $L_{rf} = \ln(\mu_r^2/\mu_f^2)$, and re-expand the formulas to NNLO in powers of $\alpha_s(\mu_r)$.

Although PIM and 1PI kinematics are used to describe different differential distributions, one can always integrate over the distributions to obtain the total hadronic cross section or the partonic scaling functions f_{ij} using the two approaches. Since the two kinematics both account for soft gluon emission, they give rise to the same results for f_{ij} in the limit $\hat{s} \rightarrow 4m_t^2$. When \hat{s} deviates from $4m_t^2$, the difference between the two kinematics is formally subleading in ξ . However, the numerical differences may be visible or even significant if \hat{s} is much larger than $4m_t^2$. In [35], we have shown that the effective-theory predictions for the total cross section from the two kinematics actually agree quite well, as long as the exact dependence on the energy of soft gluon radiation is kept in the factorization formula (5).¹ The differences between these two kinematics can be regarded as another source of theoretical uncertainty, namely that due to power corrections to the soft limit, and used along with scale uncertainties to estimate the total uncertainty associated with the calculation of the total cross section at NLO+NNLL or approximate NNLO. In the following sections we define our procedure for combining the two types of uncertainties and give our final results for the total cross section as a function of the top-quark mass in the pole, $\overline{\text{MS}}$, and 1S schemes.

III. CROSS SECTIONS IN THE POLE SCHEME

We now specify our method for estimating the theoretical uncertainties from scale variations, PDF variations and variations of α_s in the calculation of the top-pair production cross section, using either PIM or 1PI kinematics. We

¹ Previous calculations expanded $E_s(\xi)$ in the limit $z \rightarrow 1$ for PIM kinematics and $s_4 \rightarrow 0$ for 1PI kinematics, in which case the differences between the two types of kinematics become much larger.

then define a procedure for combining the results obtained in these two schemes. By default, the pole mass m_t is used in the calculation of the partonic cross sections. We comment later on alternative schemes for defining the top-quark mass.

To estimate the uncertainties associated with scale variations, we view the cross section as a function of the renormalization and factorization scales, which by default are chosen as $\mu_f = \mu_r = m_t$. We then consider two methods of scale variations: correlated variations with $\mu_f = \mu_r$ varied up and down by a factor of two from the default value, and independent variations of μ_f and μ_r by factors of two, with the uncertainties added in quadrature. We use as our final answer the larger uncertainty from these two methods.

To combine the results from PIM and 1PI kinematics, we first compute the cross sections and scale uncertainties in the PIM and 1PI schemes separately, and obtain six quantities $\sigma_{\text{PIM}}, \Delta\sigma_{\text{PIM}}^+, \Delta\sigma_{\text{PIM}}^-, \sigma_{\text{1PI}}, \Delta\sigma_{\text{1PI}}^+, \Delta\sigma_{\text{1PI}}^-$. The central value and perturbative uncertainties for the combined results are then determined by

$$\begin{aligned}\sigma &= \frac{1}{2}(\sigma_{\text{PIM}} + \sigma_{\text{1PI}}), \\ \Delta\sigma^+ &= \max(\sigma_{\text{PIM}} + \Delta\sigma_{\text{PIM}}^+, \sigma_{\text{1PI}} + \Delta\sigma_{\text{1PI}}^+) - \sigma, \\ \Delta\sigma^- &= \min(\sigma_{\text{PIM}} + \Delta\sigma_{\text{PIM}}^-, \sigma_{\text{1PI}} + \Delta\sigma_{\text{1PI}}^-) - \sigma.\end{aligned}\tag{9}$$

In this way, the central value is the average of the two, and the perturbative uncertainties reflect both the variation of the scales and the difference between the two types of kinematics. The PDF uncertainties are estimated as usual by evaluating the average of the 1PI and PIM results using the PDF error sets at a particular confidence level.

We quote in Table I the approximate NNLO predictions obtained with the above procedure at $m_t = 173.1$ GeV, using the 90% CL sets of the MSTW2008 NNLO PDFs [43]. To investigate the convergence of the perturbative series, we also list the LO and NLO results, obtained using MSTW2008 LO and NLO PDFs, respectively. In the pole scheme, the scale uncertainties are generally determined by the correlated scale variations with $\mu_r = \mu_f$. The one exception is the upper error at the Tevatron, which is instead determined by the independent variations of μ_r and μ_f added in quadrature. Even though the perturbative uncertainty in the approximate NNLO result includes both scale variations and an estimate of power corrections to the soft limit through the difference of 1PI and PIM kinematics, it is still reduced compared to that in the NLO calculation, which by definition is due only to scale variations. We note that the central value and uncertainties of the approximate NNLO results are well contained within the uncertainty range predicted by the NLO results, so that the perturbative series to this order is well behaved in the pole scheme. The NNLO results are also within the uncertainties of the LO calculation, although the NLO results are slightly higher than the LO ones in the case of the LHC.

	Tevatron		LHC7		LHC14	
	MSTW	CTEQ	MSTW	CTEQ	MSTW	CTEQ
LO	$6.66^{+2.95+(0.34)}_{-1.87-(0.27)}$	$5.45^{+2.16+0.33(0.29)}_{-1.42-0.27(0.24)}$	$122^{+49+(6)}_{-32-(7)}$	$100^{+35+9(7)}_{-24-8(7)}$	$681^{+228+(26)}_{-159-(34)}$	$552^{+157+25(18)}_{-115-25(19)}$
NLO	$6.72^{+0.41+0.47(0.37)}_{-0.76-0.45(0.24)}$	$6.77^{+0.40+0.50(0.43)}_{-0.74-0.40(0.34)}$	$159^{+20+14(8)}_{-21-13(9)}$	$148^{+18+13(11)}_{-19-12(10)}$	$889^{+107+66(31)}_{-106-58(32)}$	$829^{+97+41(27)}_{-96-40(28)}$
NNLO approx.	$6.63^{+0.07+0.63(0.33)}_{-0.41-0.48(0.25)}$	$6.91^{+0.09+0.53(0.46)}_{-0.44-0.43(0.36)}$	$155^{+8+14(8)}_{-9-14(9)}$	$153^{+8+13(11)}_{-8-12(10)}$	$855^{+52+60(30)}_{-38-59(31)}$	$842^{+51+40(26)}_{-37-40(28)}$

TABLE I: Total cross sections in pb for $m_t = 173.1$ GeV with MSTW2008 and CTEQ6.6 PDFs. The first error results from the perturbative uncertainty from both scale variations and the difference between PIM and 1PI kinematics, the second one accounts for the combined PDFs+ α_s uncertainty. The numbers in parenthesis show the PDF uncertainty only.

For comparison, we also include the results using CTEQ6.6 PDFs [44] in Table I. Since the CTEQ PDFs are based on a NLO fit, the same set is used at LO, NLO and approximate NNLO. The statements based on the analysis with MSTW PDFs above, including those concerning the moderate size of the NNLO corrections, are also true for the analysis with CTEQ PDFs. In this case, however, the LO results at the LHC are significantly lower than the NLO and NNLO results. To a certain extent, this shows the potential benefit of switching PDFs as appropriate to the order of perturbation theory. On the other hand, LO calculations are usually considered unreliable, so the more important observation for the perturbative convergence is the modest size of the NNLO correction.

The perturbative uncertainties in the approximate NNLO predictions are about the same size at both the Tevatron and the LHC. An additional source of uncertainty is related to the experimental value of $\alpha_s(M_Z)$ (where M_Z denotes the Z -boson mass), which is an input parameter for the running of the strong coupling constant. We estimate this uncertainty in combination with the PDF one by employing the method proposed in [45, 46]. Table I shows that the uncertainty on $\alpha_s(M_Z)$ adds an error of $\pm(3-4)\%$ to the pair-production cross section when the calculation is carried out with MSTW2008 PDFs. The error is somewhat smaller, $\pm(1-2)\%$, when CTEQ6.6 PDFs are used. The reason is that CTEQ6.6 assigns a 90% CL error of ± 0.002 to $\alpha_s(M_Z)$, while for MSTW2008 it is ± 0.003 . One can conclude

that the $\alpha_s(M_Z)$ induced uncertainty is of the same order of magnitude as the perturbative and PDF uncertainties, and should not be neglected.

For an extraction of the top-quark mass through a comparison with the experimental cross section, we also provide our results as a function of m_t . We parametrize the mass dependence of the approximate NNLO cross section using the simple polynomial fit

$$\sigma(m_t) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4, \quad (10)$$

where $x = m_t/\text{GeV} - 173$, and c_i are fit coefficients which depend on the collider and the PDF set. The results for the fit coefficients including upper and lower errors due to perturbative uncertainties are shown in Table II, again using MSTW2008 NNLO PDFs. A *Mathematica* implementation of the fit coefficients can be found with the electronic version of this letter, where the combined PDF and α_s uncertainties as well as the fit coefficients using CTEQ6.6 PDFs are also included. These fits reproduce the approximate NNLO calculations to 1 permille or better in the range $m_t \in [150, 180]$ GeV. For simplicity the uncertainties on fit coefficients are not displayed in Table II. When these uncertainties are measured in percent of the central value of the cross section, they appear to be roughly independent of m_t in the range $m_t \in [150, 180]$ GeV, differing by no more than a percent from those at $m_t = 173.1$ GeV shown in Table I.

		c_0 [pb]	c_1 [pb]	c_2 [pb]	c_3 [pb]	c_4 [pb]
Tevatron	σ	6.64792×10^0	-2.07262×10^{-1}	3.61739×10^{-3}	-4.30451×10^{-5}	8.94347×10^{-7}
	$\sigma + \Delta\sigma^+$	6.72257×10^0	-2.09199×10^{-1}	3.62959×10^{-3}	-5.00960×10^{-5}	6.99427×10^{-7}
	$\sigma + \Delta\sigma^-$	6.23323×10^0	-1.94555×10^{-1}	3.40149×10^{-3}	-4.03465×10^{-5}	8.17661×10^{-7}
LHC7	σ	1.55546×10^2	-4.66554×10^0	8.07632×10^{-2}	-9.93138×10^{-4}	1.75303×10^{-5}
	$\sigma + \Delta\sigma^+$	1.63325×10^2	-4.92409×10^0	8.50408×10^{-2}	-1.13686×10^{-3}	1.66808×10^{-5}
	$\sigma + \Delta\sigma^-$	1.46870×10^2	-4.40296×10^0	7.51759×10^{-2}	-1.03349×10^{-3}	1.48145×10^{-5}
LHC14	σ	8.57671×10^2	-2.29929×10^1	3.65310×10^{-1}	-4.02700×10^{-3}	7.41522×10^{-5}
	$\sigma + \Delta\sigma^+$	9.08856×10^2	-2.44668×10^1	3.89564×10^{-1}	-4.48258×10^{-3}	8.14287×10^{-5}
	$\sigma + \Delta\sigma^-$	8.19550×10^2	-2.20396×10^1	3.54379×10^{-1}	-3.81968×10^{-3}	6.25216×10^{-5}

TABLE II: Fit coefficients in (10) for the total cross sections with perturbative uncertainties at approximate NNLO, using MSTW2008 NNLO PDFs.

IV. CROSS SECTIONS IN THE $\overline{\text{MS}}$ AND 1S SCHEMES

It is well-known that the pole mass of a quark cannot be defined unambiguously in QCD due to confinement; the perturbatively defined pole mass is sensitive to long-distance physics and suffers from renormalon ambiguities of order Λ_{QCD} [47, 48]. In perturbative calculations, the renormalon ambiguity is associated with large higher-order corrections to the pole mass, and thus to any observable calculated in this scheme. Therefore, it is worth investigating short-distance mass definitions which are free from these shortcomings. In this section, we analyze the cross section as a function of the running top-quark mass defined in the $\overline{\text{MS}}$ scheme, and of the threshold top-quark mass defined in the 1S scheme [37].

It is possible to calculate the cross section using the $\overline{\text{MS}}$ mass from the beginning, by performing mass renormalization in that scheme. However, since we already have the cross section in the pole scheme, it is simpler to convert from one scheme to another using the perturbative relation between the pole mass and $\overline{\text{MS}}$ mass. This relation is currently known to three-loop order [49]. To perform the conversion to the $\overline{\text{MS}}$ scheme, we take that result for QCD with five active flavors and write it in the form

$$m_t = \overline{m}(\bar{\mu}) \left[1 + \frac{\alpha_s(\mu_r)}{\pi} d^{(1)} + \frac{\alpha_s^2(\mu_r)}{\pi^2} d^{(2)} + \mathcal{O}(\alpha_s^3) \right], \quad (11)$$

where

$$d^{(1)} = \frac{4}{3} + L_m, \quad d^{(2)} = 8.23656 + \frac{379}{72} L_m + \frac{37}{24} L_m^2 + \frac{23}{12} d^{(1)} L_r, \quad (12)$$

with $L_m = \ln(\bar{\mu}^2/\bar{m}^2(\bar{\mu}))$ and $L_r = \ln(\mu_r^2/\bar{\mu}^2)$. We then decompose the NNLO cross section in the pole scheme as

$$\sigma_{\text{NNLO}}(m_t) = \left[\frac{\alpha_s(\mu_r)}{\pi} \right]^2 \sigma^{(0)}(m_t, \mu_r) + \left[\frac{\alpha_s(\mu_r)}{\pi} \right]^3 \sigma^{(1)}(m_t, \mu_r) + \left[\frac{\alpha_s(\mu_r)}{\pi} \right]^4 \sigma^{(2)}(m_t, \mu_r), \quad (13)$$

eliminate m_t through the relation (11), and re-expand the result in powers of $\alpha_s(\mu_r)$. The resulting cross section in the $\overline{\text{MS}}$ scheme can be written as

$$\bar{\sigma}_{\text{NNLO}}(\bar{m}) = \left[\frac{\alpha_s(\bar{\mu}_r)}{\pi} \right]^2 \bar{\sigma}^{(0)}(\bar{m}(\bar{\mu}), \bar{\mu}, \mu_r) + \left[\frac{\alpha_s(\bar{\mu}_r)}{\pi} \right]^3 \bar{\sigma}^{(1)}(\bar{m}(\bar{\mu}), \bar{\mu}, \mu_r) + \left[\frac{\alpha_s(\bar{\mu}_r)}{\pi} \right]^4 \bar{\sigma}^{(2)}(\bar{m}(\bar{\mu}), \bar{\mu}, \mu_r), \quad (14)$$

where

$$\begin{aligned} \bar{\sigma}^{(0)}(\bar{m}(\bar{\mu}), \bar{\mu}, \mu_r) &= \sigma^{(0)}(\bar{m}(\bar{\mu}), \mu_r), \\ \bar{\sigma}^{(1)}(\bar{m}(\bar{\mu}), \bar{\mu}, \mu_r) &= \sigma^{(1)}(\bar{m}(\bar{\mu}), \mu_r) + \bar{m}(\bar{\mu}) d^{(1)} \left[\frac{d\sigma^{(0)}(m_t, \mu_r)}{dm_t} \right]_{m_t=\bar{m}(\bar{\mu})}, \\ \bar{\sigma}^{(2)}(\bar{m}(\bar{\mu}), \bar{\mu}, \mu_r) &= \sigma^{(2)}(\bar{m}(\bar{\mu}), \mu_r) \\ &\quad + \bar{m}(\bar{\mu}) \left[d^{(1)} \frac{d\sigma^{(1)}(m_t, \mu_r)}{dm_t} + d^{(2)} \frac{d\sigma^{(0)}(m_t, \mu_r)}{dm_t} + \frac{(d^{(1)})^2 \bar{m}(\bar{\mu})}{2} \frac{d^2 \sigma^{(0)}(m_t, \mu_r)}{dm_t^2} \right]_{m_t=\bar{m}(\bar{\mu})}. \end{aligned} \quad (15)$$

The derivatives can be taken either at the level of the hadronic cross section, using fits such as the one in (10), or at the level of the differential cross section before carrying out the phase-space integrations. We have checked our calculations by verifying the agreement between the two methods. We note that our method of converting results from the pole scheme to the $\overline{\text{MS}}$ scheme is similar to that used in [3, 50]. Indeed, our approximate NNLO results in the $\overline{\text{MS}}$ scheme for the choice $\bar{\mu} = \bar{m}$ agree with those in the HATHOR program [50], apart from the piece related to the NNLO correction $\sigma^{(2)}$, which is of course different since we are not working in the $\hat{s} \rightarrow 4m_t^2$ limit of the partonic cross section.

Our procedure for combining the results from 1PI and PIM kinematics in the $\overline{\text{MS}}$ scheme is analogous to that for the pole scheme described above. In the present case, we use by default $\mu_f = \mu_r = \bar{m}(\bar{m})$. We must also specify the scale in the running top-quark mass, for which we use $\bar{\mu} = \bar{m}$.² We provide results for the cross sections as a function of $\bar{m}(\bar{m})$ using the fit

$$\sigma(\bar{m}) = \bar{c}_0 + \bar{c}_1 \bar{x} + \bar{c}_2 \bar{x}^2 + \bar{c}_3 \bar{x}^3 + \bar{c}_4 \bar{x}^4, \quad (16)$$

where $\bar{x} = \bar{m}/\text{GeV} - 164$. The fit coefficients for the different colliders using the MSTW2008 NNLO PDFs can be found in Table III; those including combined PDF and α_s uncertainties also with CTEQ6.6 PDFs are included in the *Mathematica* notebook mentioned above.

		\bar{c}_0 [pb]	\bar{c}_1 [pb]	\bar{c}_2 [pb]	\bar{c}_3 [pb]	\bar{c}_4 [pb]
Tevatron	σ	6.66715×10^0	-2.17800×10^{-1}	3.95994×10^{-3}	-5.14404×10^{-5}	1.09983×10^{-6}
	$\sigma + \Delta\sigma^+$	6.77748×10^0	-2.21151×10^{-1}	4.04717×10^{-3}	-5.09432×10^{-5}	1.11678×10^{-6}
	$\sigma + \Delta\sigma^-$	6.26205×10^0	-2.05259×10^{-1}	3.81108×10^{-3}	-3.88796×10^{-5}	1.28549×10^{-6}
LHC7	σ	1.57441×10^2	-4.94191×10^0	9.00990×10^{-2}	-9.38583×10^{-4}	2.97762×10^{-5}
	$\sigma + \Delta\sigma^+$	1.66413×10^2	-5.20036×10^0	9.48216×10^{-2}	-1.04597×10^{-3}	2.81345×10^{-5}
	$\sigma + \Delta\sigma^-$	1.48389×10^2	-4.62721×10^0	8.26628×10^{-2}	-1.06324×10^{-3}	2.22025×10^{-5}
LHC14	σ	8.64542×10^2	-2.42364×10^1	3.98093×10^{-1}	-4.89960×10^{-3}	8.44709×10^{-5}
	$\sigma + \Delta\sigma^+$	9.20794×10^2	-2.59880×10^1	4.34218×10^{-1}	-5.00638×10^{-3}	1.02994×10^{-4}
	$\sigma + \Delta\sigma^-$	8.03504×10^2	-2.21450×10^1	3.60816×10^{-1}	-4.19956×10^{-3}	7.84995×10^{-5}

TABLE III: Fit coefficients (16) for the cross section with perturbative uncertainties at approximate NNLO in the $\overline{\text{MS}}$ scheme, using MSTW2008 NNLO PDFs.

² Variations of $\bar{\mu}$ around values close to \bar{m} , which would correspond to sampling over different mass definitions, could potentially be used as an additional means of estimating systematic uncertainties. However, a numerical analysis shows that our approximate NNLO results are very stable for variations of $\bar{\mu}$ around the default value.

The results for $\overline{m}(\overline{m}) = 164.1$ GeV, which corresponds to $m_t = 173.1$ GeV when using the two-loop conversion between the pole and $\overline{\text{MS}}$ masses, are shown in Table IV for MSTW2008 and CTEQ6.6 PDFs. As in the pole scheme, we switch the order of the MSTW PDFs according to the order of perturbation theory at which we are working, while the CTEQ PDFs are the same in both cases. In the $\overline{\text{MS}}$ scheme, the uncertainties from scale variations are dominated by the scheme where μ_f and μ_r are varied independently, rather than the scheme with correlated $\mu_r = \mu_f$ variations, as was the case in the pole scheme.

	Tevatron		LHC7		LHC14	
	MSTW	CTEQ	MSTW	CTEQ	MSTW	CTEQ
LO	$8.82^{+3.91+(0.44)}_{-2.48-(0.35)}$	$7.24^{+2.86+0.46(0.40)}_{-1.89-0.38(0.32)}$	$160^{+64+(8)}_{-42-(9)}$	$131^{+45+11(9)}_{-31-10(8)}$	$875^{+291+(32)}_{-204-(43)}$	$705^{+199+30(21)}_{-145-30(23)}$
NLO	$7.33^{+0.11+0.50(0.40)}_{-0.49-0.47(0.25)}$	$7.39^{+0.10+0.57(0.50)}_{-0.48-0.45(0.39)}$	$179^{+11+15(10)}_{-19-14(10)}$	$167^{+10+15(12)}_{-17-13(11)}$	$991^{+79+71(35)}_{-96-62(36)}$	$925^{+71+44(29)}_{-87-43(31)}$
NNLO approx.	$6.64^{+0.11+0.58(0.33)}_{-0.40-0.43(0.23)}$	$6.92^{+0.12+0.52(0.46)}_{-0.43-0.42(0.37)}$	$157^{+9+13(8)}_{-9-13(9)}$	$154^{+9+13(11)}_{-9-12(10)}$	$862^{+56+54(30)}_{-61-53(32)}$	$848^{+56+37(26)}_{-61-38(28)}$

TABLE IV: Total cross sections in pb in the $\overline{\text{MS}}$ scheme, for $\overline{m}(\overline{m}) = 164.1$ GeV. The first error results from the perturbative uncertainty from both scale variations and the difference between PIM and 1PI kinematics, the second one accounts for the combined PDFs+ α_s uncertainty. The numbers in parenthesis show the PDF uncertainty only.

We observe that the results obtained from the approximate NNLO formulas are quite close to those in the pole scheme shown in Table I, both in the central values and in the errors. Given this good agreement, which is roughly independent of the exact value of the top-quark mass as shown by the fits, it makes little practical difference whether one extracts the pole mass using the approximate NNLO results, and then determines the $\overline{\text{MS}}$ mass using the perturbative conversion (11), or whether one determines the $\overline{\text{MS}}$ mass directly, using the experimental results along with the fits at approximate NNLO. This statement would not be true at very high orders in perturbation theory, since the renormalon ambiguity inherent to the pole mass would lead to large corrections not present in a short-distance scheme such as the $\overline{\text{MS}}$ scheme. But given the present accuracy of perturbative calculations and experimental measurements, this does not yet appear to be an issue.

It is of course still interesting to study whether even at low orders the perturbative expansion is better behaved in the $\overline{\text{MS}}$ scheme than in the pole scheme. We observe that the perturbative uncertainties at NLO are generally smaller in the $\overline{\text{MS}}$ scheme than in the pole scheme, and that the central values are relatively higher compared to the approximate NNLO calculation. For this reason, the overlap between the NLO and approximate NNLO results is actually better in the pole scheme than in the $\overline{\text{MS}}$ scheme.³ These results differ from those obtained in the $\hat{s} \rightarrow 4m_t^2$ limit, where the approximated NNLO corrections and the perturbative uncertainties at that order are significantly smaller in the $\overline{\text{MS}}$ scheme than in the pole scheme [3].

To elaborate further on these results, we note that the re-organization of the perturbative expansion in the $\overline{\text{MS}}$ scheme compared to the pole scheme is accomplished by the terms in square brackets in (15). To understand whether these terms are expected to cancel against unphysically large corrections in the pole scheme, we note that the main source of mass dependence in the Born level cross section is due to phase-space factors: the lower limit of integration in (1), and an overall factor of $\sqrt{1 - 4m_t^2/\hat{s}}$ in the partonic cross section related to two-body phase space and multiplying the Born-level matrix element. The derivatives contained in the terms in square brackets are mainly sensitive to those sources of m_t dependence. However, the phase space of the pair production is more indicative of the pole mass than of an $\overline{\text{MS}}$ mass. Indeed, we are calculating the cross section for on-shell quarks according to the narrow width approximation. If the cross section is instead calculated in the $\overline{\text{MS}}$ scheme, the terms in the square brackets of the NLO and NNLO pieces of (15) give sizeable negative corrections, which are accounted in the pole scheme by using a numerically higher value of the mass in the LO and NLO cross sections. Since the most appropriate mass scheme for a given process is the one where the higher-order corrections are expected to be smallest on physical grounds, it does not seem to us that the $\overline{\text{MS}}$ scheme is the optimal choice for this case.

As an alternative to the $\overline{\text{MS}}$ mass, we consider the group of short-distance masses known as threshold masses [36]. At lower orders in perturbation theory, these are closer numerically to the pole mass, but they do not suffer from renormalon ambiguities at higher orders. The cross section in these schemes can be easily calculated from the pole-scheme results, using an analogous procedure to the $\overline{\text{MS}}$ scheme calculation. It is evident that at approximate NNLO the numerical difference between these results and the $\overline{\text{MS}}$ and pole-scheme results will be quite small once the numerical value of the mass is adjusted appropriately, but we nonetheless illustrate this with a specific example.

³ The overlap between LO and NLO is worse at the Tevatron and improved at the LHC compared to the pole scheme, but as mentioned earlier we consider the more important issue the overlap between the NLO and approximate NNLO results.

In particular, we consider the cross section as a function of the 1S mass introduced in [37]. The 1S mass is defined through the perturbative contribution to the mass of a hypothetical $n = 1$, 3S_1 toponium bound state. To perform the conversion to this scheme, we write its relation with the pole mass in the form [37]

$$m_t = m_t^{\text{1S}} \left\{ 1 + \frac{\alpha_s(\mu_r)}{\pi} \frac{2}{9} \pi \alpha_s(\mu_r) + \left(\frac{\alpha_s(\mu_r)}{\pi} \right)^2 \left[\frac{2}{9} \pi \alpha_s(\mu_r) \left(\frac{23}{3} \ln \frac{3\mu_r}{4\alpha_s(\mu_r)m_t^{\text{1S}}} + \frac{181}{18} + \frac{2}{9} \pi \alpha_s(\mu_r) \right) \right] + \mathcal{O} \left(\frac{\alpha_s^3}{\pi^3} \right) \right\}, \quad (17)$$

and follow the same procedure as for the $\overline{\text{MS}}$ scheme calculation with the appropriate replacements, cf. (11). Note that in the above relation $\pi \alpha_s$ is counted as $\mathcal{O}(1)$ and is not expanded. The results are listed in Table V for the value $m_t^{\text{1S}} = 172.3$ GeV, which corresponds to a pole mass of $m_t = 173.1$ GeV using the two-loop conversion above. The approximate NNLO results in this scheme are very similar to those in the pole and $\overline{\text{MS}}$ schemes, but the moderate size of the NNLO correction is more indicative of the pole scheme than of the $\overline{\text{MS}}$ scheme. This leads us to conclude once again that although at yet higher orders in perturbation theory the pole mass would be disfavored, at approximate NNLO accuracy this is not yet a problem.

	Tevatron		LHC7		LHC14	
	MSTW	CTEQ	MSTW	CTEQ	MSTW	CTEQ
LO	$6.83^{+3.02+(0.35)}_{-1.92-(0.28)}$	$5.59^{+2.21+0.34(0.30)}_{-1.46-0.28(0.24)}$	$124^{+50+(6)}_{-33-(7)}$	$103^{+35+9(7)}_{-24-8(7)}$	$696^{+223+(26)}_{-163-(34)}$	$564^{+160+25(18)}_{-117-25(19)}$
NLO	$6.82^{+0.39+0.48(0.38)}_{-0.75-0.46(0.24)}$	$6.87^{+0.38+0.51(0.44)}_{-0.73-0.41(0.34)}$	$162^{+19+14(9)}_{-21-13(9)}$	$150^{+17+14(11)}_{-19-12(10)}$	$902^{+106+66(32)}_{-106-59(33)}$	$841^{+96+41(27)}_{-96-40(29)}$
NNLO approx.	$6.65^{+0.06+0.63(0.32)}_{-0.38-0.47(0.24)}$	$6.93^{+0.08+0.54(0.47)}_{-0.40-0.42(0.36)}$	$156^{+7+14(9)}_{-8-14(8)}$	$154^{+7+13(11)}_{-8-12(10)}$	$859^{+47+59(30)}_{-35-58(32)}$	$846^{+46+39(25)}_{-35-40(29)}$

TABLE V: Total cross sections in pb in the 1S scheme, for $m_t^{\text{1S}} = 172.3$ GeV. The first error results from the perturbative uncertainty from both scale variations and the difference between PIM and 1PI kinematics, the second one accounts for the combined PDFs+ α_s uncertainty. The numbers in parenthesis show the PDF uncertainty only.

V. CONCLUSIONS

We have presented predictions for the total inclusive cross section for top-quark pairs at hadron colliders at approximate NNLO in QCD. Our calculations are based on soft gluon resummation to NNLL order in PIM and 1PI kinematics, carried out within the context of effective field theory. They represent the state-of-the-art, combining all knowledge presently available about higher-order QCD corrections to the production cross section. The perturbative uncertainties associated with our results are estimated in two ways: through the standard method of variations of factorization and renormalization scales, and also through the difference between the two types of kinematics. The latter gives a means of estimating the size of perturbative power corrections to the soft limits in which the approximate NNLO formulas are derived. The results presented here consolidate those previously presented in [33, 35]. We have also provided a computer program which calculates the total cross section within our approach.

The total production cross section can be used along with experimental measurements to extract the top-quark mass. An advantage of such extractions is that the theory calculations are carried out in a well-defined renormalization scheme for m_t . For very precise extractions of the top-quark mass the pole mass is disfavored, because it is only defined up to a renormalon ambiguity of order Λ_{QCD} . In practice, however, we have not observed a poor convergence of the perturbative series up to NNLO in the pole scheme compared to the $\overline{\text{MS}}$ scheme, and pointed out that the group of short-distance masses known as threshold masses may be equally appropriate. We have provided numerical fits of our results as a function of the mass in both the pole and $\overline{\text{MS}}$ schemes, including perturbative and PDF uncertainties, in addition to those from the strong coupling constant, which are non-negligible at this level of accuracy.

The results presented in this letter can be used directly by the experimental collaborations at the Tevatron and LHC in top-quark mass measurements from the corresponding production cross sections.

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- [1] H. Flächer, M. Goebel, J. Haller, A. Höcker, K. Monig, J. Stelzer, Eur. Phys. J. **C60**, 543-583 (2009). [arXiv:0811.0009 [hep-ph]].
 - [2] V. M. Abazov *et al.* [D0 Collaboration], [arXiv:1104.2887 [hep-ex]].
 - [3] U. Langenfeld, S. Moch, P. Uwer, Phys. Rev. **D80**, 054009 (2009). [arXiv:0906.5273 [hep-ph]].
 - [4] J. C. Collins, D. E. Soper, G. F. Sterman, Adv. Ser. Direct. High Energy Phys. **5**, 1-91 (1988). [hep-ph/0409313].
 - [5] P. Nason, S. Dawson, R. K. Ellis, Nucl. Phys. **B303**, 607 (1988).
 - [6] W. Beenakker, H. Kuijf, W. L. van Neerven, J. Smith, Phys. Rev. **D40**, 54-82 (1989).
 - [7] W. Beenakker, W. L. van Neerven, R. Meng, G. A. Schuler, J. Smith, Nucl. Phys. **B351**, 507-560 (1991).
 - [8] M. Czakon, A. Mitov, Nucl. Phys. **B824**, 111-135 (2010). [arXiv:0811.4119 [hep-ph]].
 - [9] M. Czakon, A. Mitov, S. Moch, Phys. Lett. **B651**, 147-159 (2007). [arXiv:0705.1975 [hep-ph]].
 - [10] M. Czakon, A. Mitov, S. Moch, Nucl. Phys. **B798**, 210-250 (2008). [arXiv:0707.4139 [hep-ph]].
 - [11] M. Czakon, Phys. Lett. **B664**, 307-314 (2008). [arXiv:0803.1400 [hep-ph]].
 - [12] R. Bonciani, A. Ferroglia, T. Gehrmann, D. Maitre, C. Studerus, JHEP **0807**, 129 (2008). [arXiv:0806.2301 [hep-ph]].
 - [13] R. Bonciani, A. Ferroglia, T. Gehrmann, C. Studerus, JHEP **0908**, 067 (2009). [arXiv:0906.3671 [hep-ph]].
 - [14] R. Bonciani, A. Ferroglia, T. Gehrmann, A. Manteuffel, C. Studerus, JHEP **1101**, 102 (2011). [arXiv:1011.6661 [hep-ph]].
 - [15] J. G. Korner, Z. Merebashvili, M. Rogal, Phys. Rev. **D77**, 094011 (2008). [arXiv:0802.0106 [hep-ph]].
 - [16] C. Anastasiou, S. M. Aybat, Phys. Rev. **D78**, 114006 (2008). [arXiv:0809.1355 [hep-ph]].
 - [17] B. Kniehl, Z. Merebashvili, J. G. Korner, M. Rogal, Phys. Rev. **D78**, 094013 (2008). [arXiv:0809.3980 [hep-ph]].
 - [18] S. Dittmaier, P. Uwer, S. Weinzierl, Phys. Rev. Lett. **98**, 262002 (2007). [hep-ph/0703120 [HEP-PH]].
 - [19] S. Dittmaier, P. Uwer, S. Weinzierl, Nucl. Phys. Proc. Suppl. **183**, 196-201 (2008). [arXiv:0807.1223 [hep-ph]].
 - [20] G. Bevilacqua, M. Czakon, C. G. Papadopoulos, M. Worek, Phys. Rev. Lett. **104**, 162002 (2010). [arXiv:1002.4009 [hep-ph]].
 - [21] K. Melnikov, M. Schulze, Nucl. Phys. **B840**, 129-159 (2010). [arXiv:1004.3284 [hep-ph]].
 - [22] M. Czakon, Phys. Lett. **B693**, 259-268 (2010). [arXiv:1005.0274 [hep-ph]].
 - [23] M. Czakon, [arXiv:1101.0642 [hep-ph]].
 - [24] C. Anastasiou, F. Herzog, A. Lazopoulos, JHEP **1103**, 038 (2011). [arXiv:1011.4867 [hep-ph]].
 - [25] G. Abelof, A. Gehrmann-De Ridder, JHEP **1104**, 063 (2011). [arXiv:1102.2443 [hep-ph]].
 - [26] W. Bernreuther, C. Bogner, O. Dekkers, [arXiv:1105.0530 [hep-ph]].
 - [27] G. F. Sterman, Nucl. Phys. **B281**, 310 (1987).
 - [28] S. Catani, L. Trentadue, Nucl. Phys. **B327**, 323 (1989).
 - [29] S. Moch, P. Uwer, Phys. Rev. **D78**, 034003 (2008). [arXiv:0804.1476 [hep-ph]].
 - [30] M. Czakon, A. Mitov, G. F. Sterman, Phys. Rev. **D80**, 074017 (2009). [arXiv:0907.1790 [hep-ph]].
 - [31] M. Beneke, M. Czakon, P. Falgari, A. Mitov, C. Schwinn, Phys. Lett. **B690**, 483-490 (2010). [arXiv:0911.5166 [hep-ph]].
 - [32] V. Ahrens, A. Ferroglia, M. Neubert, B. D. Pecjak, L. L. Yang, Phys. Lett. **B687**, 331-337 (2010). [arXiv:0912.3375 [hep-ph]].
 - [33] V. Ahrens, A. Ferroglia, M. Neubert, B. D. Pecjak, L. L. Yang, JHEP **1009**, 097 (2010). [arXiv:1003.5827 [hep-ph]].
 - [34] N. Kidonakis, Phys. Rev. **D82**, 114030 (2010). [arXiv:1009.4935 [hep-ph]].
 - [35] V. Ahrens, A. Ferroglia, M. Neubert, B. D. Pecjak, L. L. Yang, [arXiv:1103.0550 [hep-ph]].
 - [36] A. H. Hoang, M. Beneke, K. Melnikov, T. Nagano, A. Ota, A. A. Penin, A. A. Pivovarov, A. Signer *et al.*, Eur. Phys. J. direct **C2**, 1 (2000). [hep-ph/0001286].
 - [37] A. H. Hoang, T. Teubner, Phys. Rev. **D60**, 114027 (1999). [hep-ph/9904468].
 - [38] C. W. Bauer, S. Fleming, D. Pirjol, I. W. Stewart, Phys. Rev. **D63**, 114020 (2001). [hep-ph/0011336].
 - [39] C. W. Bauer, D. Pirjol, I. W. Stewart, Phys. Rev. **D65**, 054022 (2002). [hep-ph/0109045].
 - [40] M. Beneke, A. P. Chapovsky, M. Diehl, T. Feldmann, Nucl. Phys. **B643**, 431-476 (2002). [hep-ph/0206152].
 - [41] T. Becher, M. Neubert, Phys. Rev. Lett. **97**, 082001 (2006). [hep-ph/0605050].
 - [42] A. Ferroglia, M. Neubert, B. D. Pecjak, L. L. Yang, JHEP **0911**, 062 (2009). [arXiv:0908.3676 [hep-ph]].
 - [43] A. D. Martin, W. J. Stirling, R. S. Thorne, G. Watt, Eur. Phys. J. **C63**, 189-285 (2009). [arXiv:0901.0002 [hep-ph]].
 - [44] P. M. Nadolsky, H. -L. Lai, Q. -H. Cao, J. Huston, J. Pumplin, D. Stump, W. -K. Tung, C. -P. Yuan, Phys. Rev. **D78**, 013004 (2008). [arXiv:0802.0007 [hep-ph]].
 - [45] A. D. Martin, W. J. Stirling, R. S. Thorne, G. Watt, Eur. Phys. J. **C64**, 653-680 (2009). [arXiv:0905.3531 [hep-ph]].
 - [46] H. -L. Lai, J. Huston, Z. Li, P. Nadolsky, J. Pumplin, D. Stump, C. -P. Yuan, Phys. Rev. **D82**, 054021 (2010). [arXiv:1004.4624 [hep-ph]].
 - [47] I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev, A. I. Vainshtein, Phys. Rev. **D50**, 2234-2246 (1994). [hep-ph/9402360].
 - [48] M. Beneke, V. M. Braun, Nucl. Phys. **B426**, 301-343 (1994). [hep-ph/9402364].
 - [49] K. G. Chetyrkin, M. Steinhauser, Phys. Rev. Lett. **83**, 4001-4004 (1999). [hep-ph/9907509].
 - [50] M. Aliev, H. Lacker, U. Langenfeld, S. Moch, P. Uwer, M. Wiedermann, Comput. Phys. Commun. **182**, 1034-1046 (2011). [arXiv:1007.1327 [hep-ph]].